FALSE PROOFS AND COMPANY

MATHCAMP '00

Amuse friends and family with these little gems at your next holiday party.

$\mathbb{R}$ IS COUNTABLE
PROOF BY RICHARD GOTTESMAN

The axiom of choice shows that a well-ordering can be put on the reals. So put that well-ordering on $\mathbb{R}$. This means every nonempty subset of $\mathbb{R}$ has a least element. $\mathbb{R}$ is a subset of itself, so it has a least element x_1 under this ordering. Now let's look at $\mathbb{R} - \{x_1\}$. This is also a subset of $\mathbb{R}$, so it has a least element x_2. (see where this is going?) Continue to construct the sequence of reals in this manner. So we've put a 1-1 correspondence between $\mathbb{R}$ and $\mathbb{N}$. Therefore, $\mathbb{R}$ is countable.

$\mathbb{R}$ IS COUNTABLE
ZENO'S REVENGE

Between any two irrational numbers there is a rational number. Between any two rational numbers there is an irrational number. Thus, since we begin at zero, the real number line must look like rational - irrational - rational - irrational - rational - irrational - . . . . So just pair up each rational with its subsequent irrational neighbor. Thus we've created a 1 - 1 correspondence between the rationals and irrationals, the rationals are countable, and thus the irrationals are countable. The union of two countable sets is countable, and therefore, $\mathbb{R}$ is countable.

$[0, 1]$ IS COUNTABLE

Let's create a correspondence between $[0, 1]$ and the natural numbers. What one does is take each number, reverse its digits, and put a decimal point in front of it. For example, the natural number 146000 would be paired with .000641. In this manner, we will get all the reals between 0 and 1, and so $[0, 1]$ is countable.

π IS RATIONAL
PROOF BY DARREN STUART EMBRY

We shall prove this by doing induction on the decimal expansion of π. The statement P_n is that the n-digit approximation of π is rational. Well, obviously P_1 is true, for 3.1 is rational. Let us look at the inductive step by assuming P_n. Let us call the n-digit approximation q_n so that $q_n - \pi \leq 10^{-n-1}$. Since q_n is rational and has n digits, $q_n = \frac{M}{10^n}, 10^n q_n = M$. So let's look at a n+1-digit approximation of π. It involves taking q_n and adding another significant figure, so that $10^{n+1} q_{n+1} = 10M + k$, where k is an integer between -5 and 5. So $q_{n+1} = \frac{10M+k}{10^{n+1}}$ and thus P_{n+1}. So the induction holds, so for any N, the N-digit approximation is rational, so let N go to infinity, and thus π itself is rational. (This is also a good way to prove $\mathbb{R}$ is countable.

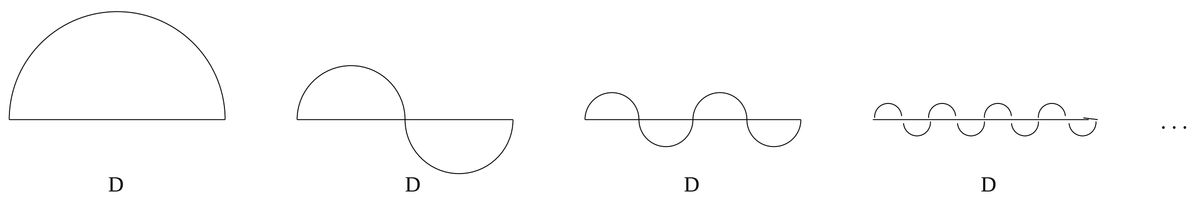
π IS ALGEBRAIC

Consider the unit circle; it has area π and its diameter is 2. Thus, it has an average width of $\frac{pi}{2}$. (This is something one sees in calculus – average width (or height, or cross-section, or whatever) is defined so that if one takes a rectangle having the same length and area, the rectangle's width is the average width. You can see a picture of where the average width occurs below.

Now consider the unit sphere; it has volume $\frac{4}{3}\pi$. As above, one notes that the diameter of the sphere is 2, so that the area of the average disk is $\frac{2}{3}\pi$. This average disk thus has a diameter of $\frac{2\sqrt{6}}{3}$ (get this from πr^2). Well, as we said before, the average width is $\frac{\pi}{2}$, so if we set this to the diameter of the average disk, we find that $\pi = \frac{4\sqrt{6}}{3}$. Thus, π is algebraic.

Date: July 2000.

$\pi = 2$
A geometrical proof



Look at the figures above. In each case, there's a line segment of length D going through the figure. In the first case, we have one half-circle with diameter D, so its length is $\frac{\pi D}{2}$. In the second case, there are 2 half-circles, each with diameter $\frac{D}{2}$, so the total arc length is still $\frac{\pi D}{2}$. So on step n, we have 2^n half-circles with diameter $\frac{\pi D}{2^n}$ each, and the total arc length is always $\frac{\pi D}{2}$. So if we let $n \to \infty$, the arc length of the subsequent figure must be $\frac{\pi D}{2}$. However, this figure, in the limit, is approaching the original line segment of length D. So $D = \frac{\pi D}{2}$, and therefore $\pi = 2$.

1 is the largest number
It's lonely at the top

Let's break up $\mathbb{R}$ into sections to look for the largest number. If we look at $(0, 1)$, we note that if we take any number x in there and look at $\frac{1}{x}$, the reciprocal is bigger than the original number. So nothing in (0,1) can be the largest number. However, 1 is the same as its reciprocal, so it's still in the running.

If we look at a negative real x and compare it to $-x$, we note the negative of the number is larger, so none of the negative numbers can be the largest. However, $1 > -1$, so it's hanging in there.

If we look at any real $x > 1$ and compare it to its square x^2, we note the square is larger than the original number, so no $n > 1$ can be the largest number. $1 = 1^2$, so it's got the rest beat.

There's only two numbers left, 0 and 1. But $1 > 0$, so that leaves 1 to be the largest number.

$2 = 1$
The insanity begins

The following derivation comes from the Taylor series expansion for natural log:

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\
\text{let } x &= 1 \\
\ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \\
2\ln 2 &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \dots \\
2\ln 2 &= (2-1) - \frac{1}{2} + (\frac{2}{3} - \frac{1}{3}) - \frac{1}{4} + (\frac{2}{5} - \frac{1}{5}) - \dots \\
2\ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\
2\ln 2 &= \ln 2 \\
2 &= 1
\end{aligned}$$

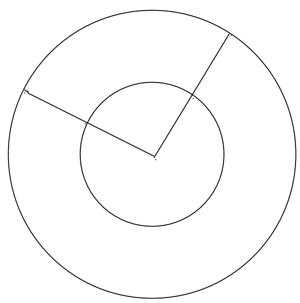
$2 = 1$
THEY JUST WANT TO BE TOGETHER

$$\begin{aligned} x &= 1 \\ x - 2 &= -1 \\ x^2 + x - 2 &= x^2 - 1 \\ (x+2)(x-1) &= (x+1)(x-1) \\ \text{strike out the common factors} & \\ x + 2 &= x + 1 \\ 2 &= 1 \end{aligned}$$

$2 = 1$
YOU CAN PROVE IT YOURSELF

Take two coins of the same denomination, the larger the better. Press one down hard with your finger so that it doesn't move, and roll the other around it, not allowing any slippage. Notice that the rolling coin will rotate *twice.* No matter how many times you try, the coin will rotate twice if you roll it around the other coin, not once. Since each point on the outer circle touches the inner circle twice, 2 = 1.

$2 = 1$
A GEOMETRIC PROOF



In the above figure, we have two concentric circles, the outer one which has twice the radius of the inner one. By drawing radii on the circles, we make a one-to-one correspondence between the points of the inner circle and the points of the outer circle. So the circles must have the same arc length. But the circumference of the outer circle is $4\pi R$ and the inner circle is $2\pi R$. Thus, $4\pi R = 2\pi R$, and thus $2 = 1$.

$2 = 1$
WHAT? YOU NEED MORE PROOF?

$$1 = \frac{i^2+3}{2} = \frac{i \cdot i + 3}{2} = \frac{\sqrt{-1}\sqrt{-1}+3}{2} = \frac{\sqrt{-1 \cdot -1}+3}{2} = \frac{\sqrt{1}+3}{2} = 2$$

$2 = 4$
OTHER NUMBERS WANT IN ON THE FUN

Consider the equation:

$$x^{x^{x^{\cdots}}} = 2$$

Since the exponent is the same as the original expression, and it's equal to 2, we can replace the exponent with 2:

$$\begin{aligned} x^2 &= 2 \\ x &= \sqrt{2} \text{ is a solution} \end{aligned}$$

Now consider the following equation, which is treated similarly to the one above:

$$\begin{aligned} x^{x^{x^{\cdots}}} &= 4 \\ x^4 &= 4 \\ x &= \sqrt{2} \text{ is a solution} \end{aligned}$$

So just substitute $\sqrt{2}$ into the $x^{x^{\cdots}}$ expression, and you get both 2 and 4. So $2 = 4$.

$0 = 1 = \frac{1}{2}$
Three's Company

Here is an interesting infinite series that has caused no end of trouble:
First way to treat it:

$$\begin{aligned} 1-1+1-1+1-1+1-1+\ldots &= (1-1)+(1-1)+(1-1)+\ldots \\ &= 0+0+0+\ldots \\ &= 0 \end{aligned}$$

Or you can look at it this way:

$$\begin{aligned} 1-1+1-1+1-1+1-\ldots &= 1+(-1+1)+(-1+1)+\ldots \\ &= 1+0+0+0+\ldots \\ &= 1 \end{aligned}$$

Or you can notice it is a geometric series, with ratio -1:

$$\begin{aligned} \sum_{k=0}^{\infty} x^k &= \frac{1}{1-x} \\ \text{In this case, } x &= 1 \\ \text{So } \sum_{k=0}^{\infty}(-1)^k &= \frac{1}{1-(-1)} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, $0 = 1 = \frac{1}{2}$.

1 = -1
VARIATION ON A THEME

Before I begin, I want to note the definition of $\delta_{j,k}$. This is called the Kronecker delta, and it is equal to 1 if $j = k$ and is equal to 0 otherwise.

$$\begin{aligned}
1 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \dots \\
&= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots \\
&= \sum_{k\geq 1} \frac{1}{k(k+1)} \\
&= \sum_{k\geq 1} \frac{k}{k+1} - \frac{k-1}{k} \\
&= \sum_{k\geq 1}\sum_{j\geq 1} \frac{k}{j}\delta_{j,k+1} - \frac{j}{k}\delta_{j,k-1} \\
&= \sum_{j\geq 1}\sum_{k\geq 1} \frac{k}{j}\delta_{j,k+1} - \frac{j}{k}\delta_{j,k-1} \\
&= \sum_{j\geq 1}\sum_{k\geq 1} \frac{k}{j}\delta_{k,j-1} - \frac{j}{k}\delta_{k,j+1} \\
&= \sum_{j\geq 1} \frac{j-1}{j} - \frac{j}{j+1} \\
&= \sum_{j\geq 1} \frac{-1}{j(j+1)} \\
&= -1
\end{aligned}$$

$e^{-2\pi} = 1$

$$\begin{aligned}
e^{i\pi} &= -1 \\
(e^{i\pi})^2 &= -1^2 \\
e^{2\pi i} &= 1 \\
(e^{2\pi i})^i &= 1^i \\
e^{-2\pi} &= 1
\end{aligned}$$

ALL NUMBERS ARE EQUAL TO 1
(WE MIGHT AS WELL GET IT OVER WITH)

We're going to do this by induction on the statement $P_n :=$ if one takes a set of n numbers, all numbers in the set are equal. Obviously P_1 is true. So let's get onto the inductive step. Suppose P_n is true. Now let us take a set of numbers of size $n+1$, and just pick an element at random and remove it from the set. We have a set of size n left, so the remaining numbers are all equal. Now, put the number you removed back in and take out a different element. Again, you will have a set of n numbers all equal. But the numbers in there had been equal to the second number you had removed. So all $n+1$ numbers are equal. The inductive step is done, so now I've proven P_n for all n.

Now let us let $n = 2$, and take the set $\{1, x\}$ where x is any number. Because P_2 is true, $x = 1$, and since x is arbitrary, we find that all numbers are equal to 1.

All natural numbers are equal
Another inductive proof

Let P_n be the statement that $1 = 2 = \ldots = n$. Obviously, P_1 is true. Now let us suppose that P_k is true.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
(k+1)^2 - (2k+1) &= k^2 \\
(k+1)^2 - (2k+1) - k(2k+1) &= k^2 - k(2k+1) \\
(k+1)^2 - (k+1)(2k+1) &= k^2 - k(2k+1) \\
\left[(k+1) - \frac{1}{2}(2k+1)\right]^2 &= \left[k - \frac{1}{2}(2k+1)\right]^2 \\
(k+1) - \frac{1}{2}(2k+1) &= k - \frac{1}{2}(2k+1) \\
k+1 &= k
\end{aligned}$$

So P_{k+1} is true, the inductive step is complete, so all natural numbers are equal.

The natural numbers don't exist
(this will make number theory easier)

Say you have a big bag of balls, and the balls are numbered consecutively: 1, 2, 3, ..., onto ∞ – one ball for each natural number. Say you stick your hand in the bag and pull out a ball at random. What is the probability that it is ball number 1? Well, there is only one ball 1, and an infinite number of balls, so the probability must be zero. But this is true for any given natural number. Then the probability of getting a natural number is the probability of getting a 1 plus the probability of getting a 2 plus ... well, you see how this goes. But each of those probabilities is zero, so the probability of getting a natural number is zero. Therefore, the natural numbers do not exist.

$\infty = -1$
I bet you didn't expect that

$$\begin{aligned}
S &= 1 + 2 + 4 + 8 + \ldots \\
&= 1 + 2(1 + 2 + 4 + \ldots) \\
&= 1 + 2S \\
S &= -1
\end{aligned}$$

However, as we add larger and larger terms, it's obvious that the sum is infinite. So $\infty = -1$.

$\infty = 0$
∞ should at least be non-negative

So, we've got all those numbered balls back and our big bag to put them in. So now you're going to do the following: at minute 1, put in balls numbered 1 until 10, and take out ball number 1 (because I like wasting time). At minute 1.5, put in balls numbered 11 to 20, and take out ball number 2. At minute 1.75, put in balls numbered 21 to 30, and take out ball number 3. At minute 1.875, put in ... well, you get the point. Obviously, all the balls will have been put in by minute 2, and have been taken out, for ball number n is removed at the n^{th} step. So there should be no balls left in the bag. However, there's an infinite number of balls in the bag, because we're always adding 9 more balls at each step. If you add up 9 an infinite number of times, you get infinity. So $\infty = 0$.

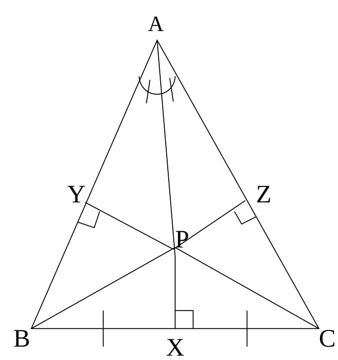
The mathematical way to lose weight

Let W = your weight, m = the weight of a mouse, and A = the average of the two weights. So:

$$\begin{aligned}
W + m &= 2A \\
(W + m)(W - m) &= 2A(W - m) \\
W^2 - m^2 &= 2AW - 2Am \\
W^2 - 2AW &= m^2 - 2Am \\
W^2 - 2AW + A^2 &= m^2 - 2Am + A^2 \\
(W - A)^2 &= (m - A)^2 \\
W - A &= m - A \\
W = m &
\end{aligned}$$

Now you weigh the same as a mouse. Aren't you happy?

All triangles are isosceles



Consider any triangle ABC (as shown above). Draw the angle bisector of A and the perpendicular bisector of side BC, whose midpoint is X. Call the intersection point of these two lines P.

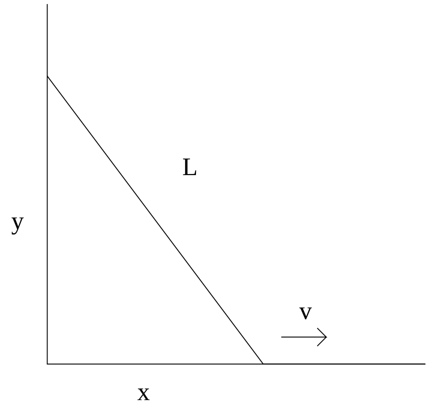
Drop perpendiculars from P to AB and AC. Call the feet of these perpendiculars Y and Z respectively. Draw PB and PC. We now have the diagram seen below.

Since AP = AP, angle YAP = angle ZAP, by side-angle-angle we have congruent triangles AYP and AZP. So now we have AY = AZ and PY = PZ.

Now we know that PB = PC, since P is on the perpendicular bisector of BC. From our previous congruent triangles, we have that PY = PZ. And since triangle PYB and triangle PZC are both right triangles, and have the hypotenuse and a leg equal, then they are congruent triangles. Thus, YB = ZC.

So we have YB=ZC and YA = ZA, therefore AB = AC and the triangle is isosceles. Since we drew triangle ABC as any old triangle, this means that all triangles are isosceles.

Ladders can fall infinitely fast
Take that, Einstein!



I know physics isn't really about proofs, but I like to think of it as a special branch of applied mathematics.

Say you have a ladder leaning up against a wall, and you're going to drag it down by pulling the bottom out from the wall. So we know the length of the ladder, we're going to pull the bottom horizontally out at a constant velocity v, and so we'll derive what the velocity of the top of ladder is.

$$\begin{aligned} y &= \sqrt{L^2 - x^2} \\ y' &= \frac{-xv}{\sqrt{L^2 - x^2}} \end{aligned}$$

So look what happens as x gets closer and closer to L. The numerator in the velocity is bounded, but the denominator is going to zero! The speed goes arbitrarily large, meaning that the top of the ladder even goes faster than the speed of light. I bet Einstein never thought about falling ladders in his gedanken experiments.

The unlucky pole vaulter

A near-sighted pole vaulter is running at relativistic speed (say at .8 the speed of light) into a barn. Her pole is 5 meters long, and the barn is also 5 meters long (these are lengths as measured in their resting frames). Unfortunately, the barn has only one door, and a very thick concrete wall opposite that door. A person standing near the barn, because of the relativistic speed, sees the vaulter and pole as having a length less than 5 meters. So this spectator knows that the pole vaulter will be inside the barn before any ugly collision occurs (he'd rather not see the mess). However, the pole vaulter sees the barn as shorter than 5 meters (because of the length contraction effect.) She thinks that the end of the pole will come out on the other side of the barn (oops, she doesn't know there's not another door) before she herself enters the barn. So we have the pole vaulter both inside and outside the barn when the collision occurs. Hmmmm... something quantum mechanical must be occurring.